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STABILITY AND STABILIZATION OF PERIODIC MOTIONS OF AUTONOMOUS SYSTEMS

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A criterion of orbital exponential stability of periodic motions of autonomous systems is obtained. The criterion is based on the method of Liapunov functions and is used together with the theory of optimal control to derive a method of stabilization of orbits.

1. Let us consider an autonomous system of differential equations (where x and f are n-vectors)

$$dx / dt = f(x) \tag{1.1}$$

Let $x = \xi(t)$ be a T-periodic solution of (1.1) different from the point of rest and γ the trajectory of this solution. We shall assume that the components of the vector f are sufficiently smooth functions in some neighborhood of the trajectory γ. For any point x sufficiently near to the trajectory γ, we can find a unique quantity $\theta(x)$ such, that $0 \leqslant \theta(x) < T$, $\xi(\theta(x))$ is the point on the trajectory γ nearest to x, and the vector $x - \xi(\theta(x))$ is orthogonal to the vector $f(\xi(\theta(x)))$.

Definition. A periodic solution $\xi(t)$ of the system (1.1) shall be called exponentially orbitally stable (EO-stable) if $\delta > 0$, $\alpha > 0$ and $K > 0$ exist such that

$$|x(t) - \xi(\theta(x(t)))| \leqslant Ke^{-\alpha(t-t_0)} |x_0 - \xi(\theta(x_0))| \tag{1.2}$$

as soon as $|x_0 - \xi(\theta(x_0))| < \delta$. In the expression (1.2) $x(t)$ denotes a solution of (1.1) emerging from the point x_0 at the initial instant of time t_0.

The sufficient conditions of the EO stability are related to the Andronov – Vitt theorem [1,2] and its analogs [3,4], and go back to Liapunov [5]. These conditions consist of the fact that the variational equations

$$\frac{dy}{dt} = F(t)y, \quad F(t) = \left\{\frac{\partial f_i}{\partial x_j}(\xi(t))\right\} \tag{1.3}$$

of the system (1.1) have, for the periodic solution $\xi(t)$, a single simple zero characteristic index, and all remaining indices have negative real parts.

A theorem given in [6] asserts that these conditions are also necessary for the EO stability of the periodic solution $\xi(t)$ of (1.1). The present paper gives a criterion of EO stability based on the method of Liapunov functions. Use of this criterion enables us to solve the problem of stabilization of periodic motion $\xi(t)$ of the system (1.1).

2. Lemma. If $v(x)$ is a sufficiently smooth function in the neighborhood of the trajectory γ, $v \geqslant 0$ and $v(\xi(\tau)) = 0$, $0 \leqslant \tau < T$, then

$$v(x) = (x - \xi(\tau))^* V(\tau)(x - \xi(\tau)) + O(|x - \xi(\tau)|^3) \tag{2.1}$$

$$V(\tau) f(\xi(\tau)) = 0 \tag{2.2}$$

$$\sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i(x) \equiv R(x) = (x - \xi(\tau))^* (F^*(\tau) V(\tau) + \tag{2.3}$$
$$V(\tau) F(\tau) + V'(\tau))(x - \xi(\tau)) + O(|x - \xi(\tau)|^3)$$
$$\left(V(\tau) = \frac{1}{2}\left\{\frac{\partial^2 v}{\partial x_i \partial x_j}(\xi(\tau))\right\}\right)$$

Here $V(\tau)$ is a T - periodic matrix and the ratios of the quantities $O(|x - \xi(\tau)|^3)$ in (2.1) and (2.3) to $|x - \xi(\tau)|^3$ are bounded uniformly in τ for small $|x - \xi(\tau)|$.

Proof. At the point of the curve γ the function $v(x)$ attains a minimum, therefore

$$\partial v / \partial x_i (\xi(\tau)) = 0, \quad i = 1, \ldots, n, \quad 0 \leqslant \tau < T \tag{2.4}$$

Using (2.4) and the Taylor formula to represent $v(x)$ in the neighborhood of the point $\xi(\tau)$, we obtain (2.1).

Let us compute the i -th coordinate of the vector $V(\tau) f(\xi(\tau))$

$$\sum_{k=1}^{n} \frac{\partial^2 v}{\partial x_i \partial x_k}(\xi(\tau)) f_k(\xi(\tau)) = \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(\frac{\partial v}{\partial v_i}\right)(\xi(\tau)) f_k(\xi(\tau)) = \frac{d}{d\tau}\left(\frac{\partial v}{\partial x_i}\right)(\xi(\tau)) \tag{2.5}$$

where the last expression represents, by virtue of (1.1), the total derivative of $\partial v / \partial x_i$. From (2.4) and (2.5) we obtain (2.2).

Next we prove the formula (2.3). We expand the function $R(x)$ into a Taylor series at the point $\xi(\tau)$. This yields

$$R(x) = R(\xi(\tau)) + \sum_{i=1}^{n} \frac{\partial R}{\partial x_i}(\xi(\tau))(x_i - \xi_i(\tau)) + \tag{2.6}$$
$$\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 R}{\partial x_i \partial x_j}(\xi(\tau))(x_i - \xi_i(\tau))(x_j - \xi_j(\tau)) + O(|x - \xi(\tau)|^3)$$

We have (see (2.4)) $R(\xi(\tau)) = 0$. Further we have

$$\frac{\partial R}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial^2 v}{\partial x_k \partial x_i} f_k + \sum_{k=1}^{n} \frac{\partial v}{\partial x_k} \frac{\partial f_k}{\partial x_i} \tag{2.7}$$

$$\frac{\partial^2 R}{\partial x_i \partial x_j} = \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right) f_k + \sum_{k=1}^{n} \frac{\partial^2 v}{\partial x_k \partial x_i} \frac{\partial f_k}{\partial x_j} + \tag{2.8}$$
$$\sum_{k=1}^{n} \frac{\partial^2 v}{\partial x_k \partial x_j} \frac{\partial f_k}{\partial x_i} + \sum_{k=1}^{n} \frac{\partial v}{\partial x_k} \frac{\partial^2 f_k}{\partial x_i \partial x_j}$$

By virtue of (2.4) and (2.2), $\partial R(\xi(\tau)) / \partial x_i = 0$. Further, when $x = \xi(\tau)$, the first sum in the right-hand side of (2.8) is equal, by virtue of (1.1), to the derivative of the function $\partial^2 v / \partial x_i \partial x_j$ with respect to τ, along the periodic solution $x = \xi(\tau)$. Therefore the sum is equal to the element of the i - th row and j - th column of the matrix $V'(\tau)$. The last sum in (2.8) vanishes when $x = \xi(\tau)$ by virtue of (2.4). As the result we obtain

$$\frac{1}{2}\left\{\frac{\partial^2 R}{\partial x_i \partial x_j}(\xi(\tau))\right\} = V'(\tau) + F^*(\tau)V(\tau) + V(\tau)F(\tau)$$

and this yields (2.3).
The uniform boundedness of the ratio mentioned in the lemma follows from the assumed smoothness, and this proves the lemma.

Let us give a graphical interpretation of the lemma. If x is a two-dimensional vector, the graph of the Liapunov function v with the lemma is concerned, represents an annular groove the base of which is the orbit $x = \xi(t)$. Figure 1 depicts a section of this groove. A graph of the function $(x - \xi(\tau))^* V(\tau)(x - \xi(\tau))$, is also drawn through the point M lying on the orbit. The function gives the corresponding approximation to the function $v(x)$ in the neighborhood of the point $M(\xi_1(\tau), \xi_2(\tau))$ and is a parabolic cylinder. This illustrates clearly the degeneracy of the matrix $V(\tau)$ and the relation (2.2).

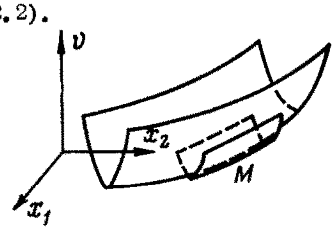

Fig. 1

Let us denote by P_f the matrix corresponding to the projection operator acting on the subspace orthogonal to the vector $f \neq 0$; $P_f = E - |f|^{-2} ff^*$ and E is a unit matrix. We shall call the quadratic form x^*Ax and the symmetric matrix A P_f-positive definite (P_f-nonnegative definite) if for any vector $x \neq 0$ orthogonal to the vector f the inequality $x^*Ax > 0$ ($x^*Ax \geqslant 0$) holds.

Theorem 1. For the EO stability of a T-periodic solution $\xi(t)$ of the system (1.1) it is necessary and sufficient that for any T-periodic matrix $C(\tau)$ and any T-periodic nonnegative function $\alpha(\tau)$ such that

$$\int_0^T \alpha(\tau)\, d\tau > 0 \qquad (2.9)$$

and the matrix $C(\tau) - \alpha(\tau)E$ is $P_{f(\xi(\tau))}$-nonnegative definite, a T-periodic, $P_{f(\xi(\tau))}$-positive definite matrix $V(\tau)$ exists such that (2.2) holds and

$$V'(\tau) + F^*(\tau)V(\tau) + V(\tau)F(\tau) = -P_{f(\xi(\tau))}C(\tau)P_{f(\xi(\tau))} \qquad (2.10)$$

Proof. Sufficiency. Let us introduce the function $v(x) = (x - \xi(\theta))^* V(\theta) \cdot (x - \xi(\theta))$. Here and henceforth $\theta = \theta(x)$. It can be proved that $\frac{1}{2}\{\partial^2 v(\xi(\tau)) / \partial x_i \partial x_j\} = V(\tau)$. From the lemma follows

$$\frac{dv}{dt} = \sum_{i=1}^n \frac{\partial v}{\partial x_i} f_i(x) = (x - \xi(\theta))^* W(\theta)(x - \xi(\theta)) + O(|x - \xi(\theta)|^3) \qquad (2.11)$$

$$(W(\tau) = V'(\tau) + F^*(\tau)V(\tau) + V(\tau)F(\tau))$$

By virtue of (2.10), (2.11) and the fact that the vectors $f(\xi(\theta))$ and $(x - \xi(\theta))$ are orthogonal, we have

$$dv / dt = -(x - \xi(\theta))^* C(\theta)(x - \xi(\theta)) + O(|x - \xi(\theta)|^3) \qquad (2.12)$$

Since the matrix $V(\theta)$ is $P_{f(\xi(\theta))}$-positive definite, we can find positive numbers m

and M such that

$$m\,(x-\xi(\theta))^*\,(x-\xi(\theta))\leqslant(x-\xi(\theta))^*V(\theta)\,(x-\xi(\theta))\leqslant M\,(x-\xi(\theta))^*\,(x-\xi(\theta)) \tag{2.13}$$

Conditions of the theorem and (2.13) together yield the inequality

$$-(x-\xi(\theta))^*\,C(\theta)\,(x-\xi(\theta))\leqslant-\alpha(\theta)\,(x-\xi(\theta))^*\,(x-\xi(\theta))\leqslant -\frac{\alpha(\theta)}{M}(x-\xi(\theta))^*\,V(\theta)\,(x-\xi(\theta))$$

which, together with (2.12), yields the following inequality for sufficiently small $|x-\xi(\theta)|$:

$$\frac{dv}{dt}\leqslant-\frac{\alpha(\theta)}{2M}v \tag{2.14}$$

The above inequality ensures, by virtue of (2.13), that $|x-\xi(\theta)|$ is small at all instants of time provided that $|x_0-\xi(\theta(x_0))|$ is small.

Using the fact that the solutions are continuous with respect to the initial parameters, the T-periodicity of the function $\alpha(\tau)$ and the condition (2.9), we obtain for small $|x_0-\xi(\theta(x_0))|$

$$\int_{rT}^{(r+1)T}\alpha(\theta(x(t)))\,dt\geqslant\alpha_0>0,\qquad r=0,1,\ldots,(x(0)=x_0) \tag{2.15}$$

Dividing both sides of (2.14) by v, integrating from 0 to t and taking into account (2.15), we obtain

$$v(t)\leqslant Kv(0)e^{-\alpha t} \tag{2.16}$$

where $K>0$ and $\alpha>0$ are some constants. Owing to (2.13), the inequality (2.16) proves the EO stability of the solution $x=\xi(t)$ of the system (1.1).

N e c e s s i t y. Let us consider the function

$$v(x)=\int_0^\infty(x(t)-\xi(\theta))^*\,C(\theta)\,(x(t)-\xi(\theta))\,dt \tag{2.17}$$
$$(x(0)=x,\ \theta=\theta(x(t)))$$

where $x(t)$ is the solution of (1.1). Function $v(x)$ satisfies the conditions of the lemma. Clearly, $V(\tau)$ is a $P_{f(\xi(\tau))}$-positive definite matrix for the function $v(x)$ of (2.17). The formula (2.11) holds for $v(x)$. On the other hand, using (2.17) we find that

$$dv\,/\,dt=-(x-\xi(\theta))^*C(\theta)\,(x-\xi(\theta)) \tag{2.18}$$

Equating (2.11) with (2.18), we obtain

$$P_{f(\xi(\tau))}W(\tau)\,P_{f(\xi(\tau))}=-P_{f(\xi(\tau))}C(\tau)\,P_{f(\xi(\tau))} \tag{2.19}$$

But from (2.2) we have

$$[V'(\tau)+V(\tau)F(\tau)]f(\xi(\tau))=0$$

and this, together with (2.2), yields

$$W(\tau)P_{f(\xi(\tau))}=W(\tau) \tag{2.20}$$

Applying the conjugation operation to both sides of (2.20) we obtain another equation which, together with (2.19) and (2.20), yield (2.10), and this completes the proof of Theorem 1.

3. The problem of stabilization of the points of rest was studied by many authors [7,8]. Let us consider the problem of stabilization of the periodic motion of the system (1.1). We introduce a system with a control

$$dx / dt = f(x) + b(\theta(x))u \tag{3.1}$$

where $b(\tau)$ is a T-periodic n-vector and u is a scalar control. We shall seek a control $u = u(x_1, \ldots, x_n)$ from the condition of minimization of the functional

$$J = \int_0^\infty [(x - \xi(\theta))^* C(\theta)(x - \xi(\theta)) + \beta(\theta) u^2]\, dt, \quad \theta = \theta(x) \tag{3.2}$$

where $C(\tau)$ satisfies the requirements of Theorem 1 and $\beta(\tau)$ is a T-periodic positive function.

The Bellman function $v^\circ(x_1, \ldots, x_n)$ of the problem (3.1), (3.2) satisfies the equation

$$\min_u \Big[\sum_{i=1}^n \frac{\partial v^\circ}{\partial x_i}(f_i(x) + b_i(\theta) u) + (x - \xi(\theta))^* C(\theta)(x - \xi(\theta)) + \beta(\theta) u^2] = 0 \tag{3.3}$$

Assuming that the function v° is sufficiently smooth, we can confirm that it satisfies the conditions of the lemma. This leads to the relations (2.1) and (2.2) where

$$V(\tau) = {}^1/_2 \{\partial^2 v^\circ(\xi(\tau)) / \partial x_i \partial x_j\}.$$

Using (3.3) we obtain

$$u^\circ = -\frac{1}{\beta(\theta)} b^*(\theta) V(\theta)(x - \xi(\theta)) + O(|x - \xi(\theta)|^2) \tag{3.4}$$

$$\begin{gathered} V'(\tau) + F^*(\tau) V(\tau) + V(\tau) F(\tau) - \\ \frac{1}{\beta(\tau)} V(\tau) b(\tau) b^*(\tau) V(\tau) = -P_{f(\xi(\tau))} C(\tau) P_{f(\xi(\tau))} \end{gathered} \tag{3.5}$$

Theorem 2. Let $P_{f(\xi(\tau))}$ - positive definite T-periodic matrix $V(\tau)$ satisfy (2.2) and (3.5). Then the T-periodic motion $x = \xi(t)$ of the system (3.1) with control

$$u(x) = -\frac{1}{\beta(\theta)} b^*(\theta) V(\theta)(x - \xi(\theta)) \tag{3.6}$$

is EO-stable. The value of the functional (3.2) under the control (3.6) is equal to $v(x) + O(|x - \xi(\theta)|^3)$ where $v(x) = (x - \xi(\theta))^* V(\theta)(x - \xi(\theta))$, and differs from its optimal value by $O(|x - \xi(\theta)|^3)$.

Proof. Equations in variations of the system (3.1) with control (3.6) have the following form for the periodic solution $x = \xi(t)$:

$$\frac{dy}{dt} = G(t) y = \left(F(t) - \frac{1}{\beta(t)} b(t) b^*(t) V(t)\right) y \tag{3.7}$$

Replacing F in (2.10) by the matrix G and using (3.5), we find

$$V'(\tau) + G^*(\tau) V(\tau) + V(\tau) G(\tau) = -P_{f(\xi(\tau))} C(\tau) P_{f(\xi(\tau))} - \frac{1}{\beta(\tau)} V(\tau) b(\tau) b^*(\tau) V(\tau) \tag{3.8}$$

Equation (3.8) proves, by virtue of Theorem 1, the EO stability of the solution

$x = \xi(t)$ of (3.1) with the control (3.6). The rest of the theorem is proved with the help of the method given in [9]. Let us consider (3.2) and the functional

$$K = J + \int_0^\infty \sum_{i=1}^n \frac{\partial v}{\partial x_i} (f_i(x) + b_i(\theta) u)\, dt$$

Clearly, if the control u in the problem of minimizing the functional J is optimal, then it will also be optimal in the problem of minimizing K and vice versa. This follows from the equality $K(x) = J(x) - v(x)$. The integrand $Q(x, u)$ of the functional K can be written with the help of the lemma in the form

$$Q(x, u) = (x - \xi(\theta))^* [C(\theta) + V'(\theta) + F^*(\theta) V(\theta) + V(\theta) F(\theta)] (x - \xi(\theta)) + \beta(\theta) u^2 + 2b^*(\theta) V(\theta)(x - \xi(\theta)) u + O(|x - \xi(\theta)|^2) u + O(|x - \xi(\theta)|^3)$$

From (3.5) follows $\quad \min_u Q(x, u) = O(|x - \xi(\theta)|^3)$

and this yields for any control

$$J(x) = K(x) + v(x) \geqslant v(x) - |O(|x - \xi(\theta)|^3)|$$

But under the control (3.6) $J = v(x) + O(|x - \xi(\theta)|^3)$. This completes the proof of Theorem 2.

4. Example. Let us consider the problem of stabilizing the motion $\xi_1 = r \cos t$, $\xi_2 = -r \sin t$ in the system

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = -x_1 + b(\theta) u \tag{4.1}$$

with the minimization of the functional

$$J = \int_0^\infty [c_1(\theta)(x_1 - \xi_1(\theta))^2 + c_2(\theta)(x_2 - \xi_2(\theta))^2 + \beta(\theta) u^2]\, dt \tag{4.2}$$

The relation (2.2) and (3.5) yield the following expressions for the elements of the matrix $V(\tau)$:

$$v_{11} = 2\lambda(\tau) \cos^2 \tau, \quad v_{12} = v_{21} = -\lambda(\tau) \sin 2\tau, \quad v_{22} = 2\lambda(\tau) \sin^2 \tau$$

where the 2π-periodic positive function $\lambda(\tau)$ satisfies the Riccati equation

$$\lambda' - \frac{2b^2(\tau)}{\beta(\tau)} \sin^2 \tau \lambda^2 + \frac{1}{2} c_1(\tau) \cos^2 \tau + \frac{1}{2} c_2(\tau) \sin^2 \tau = 0 \tag{4.3}$$

The function $v(x_1, x_2)$ and control $u(x_1, x_2)$, are nearly optimal at small $|\sqrt{x_1^2 + x_2^2} - r|$ and are, respectively,

$$v(x_1, x_2) = 2\lambda(\theta)(\sqrt{x_1^2 + x_2^2} - r)^2 \tag{4.4}$$

$$u = -2\lambda(\theta)(x_2 - r x_2 / \sqrt{x_1^2 + x_2^2}) \tag{4.5}$$

where the function $\theta(x_1, x_2)$ can be obtained from the relations

$$\cos\theta = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \quad \sin\theta = -\frac{x_2}{\sqrt{x_1^2 + x_2^2}}$$

If, for example, $b = \beta = 1$, $c_1 = 0$, $c_2 = 1$, then (4.3) yields $\lambda \equiv 1/2$. In this case the system (4.1) with control (4.5) assumes the form

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = -x_1 - (x_2 - rx_2 / \sqrt{x_1^2 + x_2^2}) \tag{4.6}$$

The solution $x_1 = r \cos t$, $x_2 = -r \sin t$ of (4.6) is self-oscillatory.

Note. The sufficient criterion of stabilizability (Theorem 3) given in [10] is incorrect. It turns out that the full controllability is insufficient to ensure the stabilizability of the systems with arbitrary noise. The noise must have certain restrictions imposed on it. For example, the following result is valid: if the system (2.2) is fully controlled and a number $\alpha > 0$ is found such that for any $D>0$ the inequality

$$\sum_{r=1}^{k} \sigma_r^* D \sigma_r < \alpha D$$

holds, then the system (1.1) can be stabilized in the quadratic mean if $\varphi_r = 0$ $(r = 1, 2, \ldots, m)$.

We note that the criterion formulated in Theorem 3 is not related to the basic content of [10], and the discussion of the remaining material does not make use of this theorem.

REFERENCES

1. Andronov A. and Vitt A., On the Liapunov stability. ZhETF, Vol.3, № 5, 1933.

2. Pontriagin L. S., Ordinary Differential Equations. (English translation), Pergamon Press, Book № 09699, 1964.

3. Coddington E. A. and Levinson N., Theory of Ordinary Differential Equations. N. Y., McGraw-Hill, 1955.

4. Demidovich B. P., Lectures in the Mathematical Theory of Stability. Moscow, "Nauka", 1967.

5. Liapunov A. M., General Problem of Stability of Motion. Moscow-Leningrad, Gostekhizdat, 1950.

6. Hartman P., Ordinary Differential Equations. N. Y., Wiley, 1964.

7. Letov A. M., Analytic construction of regulators. Avtomatika i telemekhanika, Vol.21, № 4-6, 1960, Vol.22, № 4, 1961 and Vol.23, № 11, 1962.

8. Krasovskii N. N., Problems of Stabilization of Controlled Motions. Add. to the book by I. G. Malkin: Theory of Stability of Motion. Moscow, "Nauka", 1966.

9. Mil'shtein G. N., Solving an optimal problem by the method of consecutive approximations. Avtomatika i telemekhanika, Vol.25, № 3, 1964.

10. Mil'shtein G. N. and Riashko L. B., Optimal stabilization of linear stochastic systems. PMM Vol.40, № 6, 1976.

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